Problem Set Three

Due date: Tursday March 4, 2008

1. **(10pt)** Consider a communication network represented by a graph with n nodes and m edges. You have k packets sitting at various nodes in the network. The goal is to construct a schedule that will bring all these packets to a designated node s as fast as possible (i.e. minimize the the number of steps before the last packet arrives). By "schedule" we mean a table that, for each time unit, indicates which packets will cross which edges during this time unit. The restriction is that each communication link of the network is capable of transmitting only one packet every time unit. [Hint: show a reduction to max flow in an appropriately constructed graph.]

2. **(10pt)** Suppose that after solving a maximum flow problem with integer capacities you realize that you have overestimated the capacity of an arc e by k units. Show how to reoptimize the problem in $O(km)$ time.

3. **(10pt)** Suppose that we wish to identify, from among all minimum cuts, a minimum cut containing the least number of arcs. Show how we can modify the capacities of the graph so that the minimum cut with respect to our modified capacities is the minimum cut with the least number of arcs.

4. **(20pt)** You are given a directed capacitated graph and your goal is to find max-flow in this graph. Assume that the capacities are only $1, 2$, or 3. Using the algorithms studied in class, one can solve this problem in $O(n^3)$ without using dynamic trees. Can you design an algorithm that works in $O(nm)$? [Hint: this part is easy.] Now, improve the running time beyond this bound. [Hint: Combine layered-network algorithm with Ford-Fulkerson. Show that if the network has a lot of layers, then this limits the residual flow.]

5. **(10pt)** Consider a variant of max flow where the constraints are on the nodes instead of the edges. In other words, each vertex has some capacity $cap(v)$, and no more than that amount of flow can go through that vertex. Each edge has infinite capacity. Use the normal version of max flow (i.e. with capacities on edges) to solve this problem.

6. **(10pt)**

   A police department in a small city consists of three precincts denoted p_1, p_2, and p_3. Each precinct is assigned a number of patrol cars equipped with two-way radios and first-aid equipment. The department operates with three shifts. The two tables below show the minimum and maximum number of patrol cars needed in each shift. Administrative constraints require that (1) shifts 1, 2, and 3 have, respectively, at least 10, 20, and 18 cars available; and (2) precincts p_1, p_2, and p_3 are, respectively, allocated at least 10, 14, and 13 cars. The police department wants to determine an allocation of patrol units that will meet all the requirements with the fewest possible units committed to the field. Explain how to solve this problem using max-flow techniques. (Don't give an actual numerical solution. Just give an algorithm using network flow techniques.) Any polynomial running-time bound is sufficient. Make sure you prove that your algorithm produces an optimum solution.

Minimum cars required per shift

	Shift 1	Shift 2	Shift 3
p_1	2	4	3
p_2	3	6	5
p_3	5	7	6

Maximum cars allowed per shift

	Shift 1	Shift 2	Shift 3
p_1	3	7	5
p_2	5	7	10
p_3	8	12	10